Reductive G-structures and Lie derivatives

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Abstract

Reductive G-structures on a principal bundle Q are considered. It is shown that these structures, i.e. reductive G-subbundles P of Q, admit a canonical decomposition of the pull-back vector bundle $i_P^*(TQ) \equiv P \times_Q TQ$ over P. For classical G-structures, i.e. reductive G-subbundles of the linear frame bundle, such a decomposition defines an infinitesimal canonical lift. This lift extends to a prolongation Γ-structure on P. In this general geometric framework the theory of Lie derivatives is considered. Particular emphasis is given to the morphisms which must be taken in order to state what kind of Lie derivative has to be chosen. On specializing the general theory of gauge-natural Lie derivatives of spinor fields to the case of the Kosmann lift, we recover the result originally found by Kosmann. We also show that in the case of a reductive G-structure one can introduce a "reductive Lie derivative" with respect to a certain class of generalized infinitesimal automorphisms. This differs, in general, from the gauge-natural one, and we conclude by showing that the "metric Lie derivative" introduced by Bourguignon and Gauduchon is in fact a particular kind of reductive rather than gauge-natural Lie derivative.
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1. Introduction

It has now become apparent that there has been some confusion regarding the concept of a Lie derivative of spinor fields, both in the mathematical and the physical literature.

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Lichnerowicz was the first one to give a correct definition for such an object, although with respect to infinitesimal isometries only. The local expression given by Lichnerowicz in 1963 [24] is

$$\pounds_\xi \psi := \xi^a \nabla_a \psi - \frac{1}{4} \nabla_a \xi_b \gamma^a \gamma^b \psi, \qquad (*)$$

where $\nabla_a \xi_b = \nabla_{[a} \xi_{b]}$, as ξ is assumed to be a Killing vector field.

After a first attempt to extend Lichnerowicz's definition to generic infinitesimal transformations [21], in 1972 Kosmann put forward a new definition of a Lie derivative of spinor fields in [22], her doctoral thesis under Lichnerowicz's supervision. Indeed, in her previous work she had just extended *tout court* Lichnerowicz's definition to the case of a generic vector field ξ, without antisymmetrizing $\nabla_a \xi_b$. Therefore, the local expression appearing in [21] could not be given any clear-cut geometrical meaning. The remedy was then realized to be retaining Lichnerowicz's local expression $(*)$ for a *generic* vector field ξ, but explicitly taking the antisymmetric part of $\nabla_a \xi_b$ only [22].

Several papers on the subject followed, including particularly Binz and Pferschy's [1] and Bourguignon and Gauduchon's [2]. Furthermore, among the physics community much interest has been attracted by Penrose and Rindler's definition [26], despite its being restricted to infinitesimal conformal isometries because of the (implicit) requirement that the Lie derivative commute with the isomorphism between the complexified tangent bundle and the tensor product of the spinor bundle and its complex conjugate (see [5] for a thorough discussion).

In this paper we investigate whether the definition of a Lie derivative of spinor fields can be placed in the more general framework of the theory of Lie derivatives of sections of fibred manifolds (and, more generally, of differentiable maps between two manifolds) stemming from Trautman's 1972 seminal paper [27] and further developed by Janyška and Kolář [16] (see also [20]).

A first step in this direction was already taken in [7], where Kosmann's 1972 definition was successfully placed in the framework of the theory of Lie derivatives of sections of *gauge-natural bundles* by introducing a new geometric concept, which the authors called the "Kosmann lift".

The aim of this paper is to provide a more transparent geometric explanation of the Kosmann lift and, at the same time, a generalization to reductive G-structures. Indeed, the Kosmann lift is but a *particular case* of this interesting generalization.

The structure of the paper is as follows: in Section 2 preliminary notions on principal bundles are recalled for the main purpose of fixing our notation; in Section 3 the concept of a reductive G-structure and its main properties are introduced; in Section 4 a constructive approach to gauge-natural bundles is proposed together with a number of relevant examples; in Section 4 split structures on principal bundles are considered and the notion of a generalized Kosmann lift is defined; finally, in Section 6 the general theory of Lie derivatives is applied to the context of reductive G-structures, allowing us to analyse the concept of the Lie derivative of spinor fields in all its different flavours from the most general point of view. The proofs of the results presented in this paper mainly consist of the careful application of the definitions which precede them, and therefore are mostly omitted.

2. Notation

Let M be a manifold and G a Lie group. A *principal (fibre) bundle* P over M with structure group G is obtained by attaching a copy of G to each point of M, i.e. by giving a G-manifold P, on which G acts on the right and which satisfies the following conditions:

1. The (right) action $r : P \times G \to P$ of G on P is *free*, i.e. $u \cdot a := r(u, a) = u, u \in P$, implies $a = e$, e being the unit element of G.
2. $M = P/G$ is the quotient space of P by the equivalence relation induced by G, i.e. M is the space of orbits. Moreover, the canonical projection $\pi : P \to M$ is smooth.
3. P is locally trivial, i.e. P is locally a product $U \times G$, where U is an open set in M. More precisely, there exists a diffeomorphism $\Phi : \pi^{-1}(U) \to U \times G$ such that $\Phi(u) = (\pi(u), f(u))$, where the mapping $f : \pi^{-1}(U) \to G$ is G-*equivariant*, i.e. $f(u \cdot a) = f(u) \cdot a$ for all $u \in \pi^{-1}(U), a \in G$.

A principal bundle will be denoted by $(P, M, \pi; G)$, $P(M, G)$, $\pi : P \to M$ or simply P, according to the particular context. P is called the *bundle* (or *total*) *space*, M the *base*, G the *structure group*, and π the *projection*. The closed submanifold $\pi^{-1}(x)$, $x \in M$, will be called the *fibre over* x. For any point $u \in P$, we have $\pi^{-1}(x) = u \cdot G$, where $\pi(u) = x$, and $u \cdot G$ will be called the *fibre through* u. Every fibre is diffeomorphic to G, but such a diffeomorphism depends on the chosen trivialization.

Given a manifold M and a Lie group G, the product manifold $M \times G$ is a principal bundle over M with projection $\mathrm{pr}_1 : M \times G \to M$ and structure group G, the action being given by $(x, a) \cdot b = (x, a \cdot b)$. The manifold $M \times G$ is called a *trivial principal bundle*.

A *homomorphism* of a principal bundle $P'(M', G')$ into another principal bundle $P(M, G)$ consists of a differentiable mapping $\Phi : P' \to P$ and a Lie group homomorphism $f : G' \to G$ such that $\Phi(u' \cdot a') = \Phi(u') \cdot f(a')$ for all $u' \in P', a' \in G'$. Hence, Φ maps fibres into fibres and induces a differentiable mapping $\varphi : M' \to M$ by $\varphi(x') = \pi(\Phi(u'))$, u' being an arbitrary point over x'. A homomorphism $\Phi : P' \to P$ is called an *embedding* if $\varphi : M' \to M$ is an embedding and $f : G' \to G$ is injective. In such a case, we can identify P' with $\Phi(P')$, G' with $f(G')$ and M' with $\varphi(M')$, and P' is said to be a *subbundle* of P. If $M' = M$ and $\varphi = \mathrm{id}_M$, P' is called a *reduced subbundle* or a *reduction* of P, and we also say that G "reduces" to the subgroup G'.

A homomorphism $\Phi : P' \to P$ is called an *isomorphism* if there exists a homomorphism of principal bundles $\Psi : P \to P'$ such that $\Psi \circ \Phi = \mathrm{id}_{P'}$ and $\Phi \circ \Psi = \mathrm{id}_P$.

3. Reductive G-structures and their prolongations

Definition 3.1. Let H be a Lie group and G a Lie subgroup of H. Denote by $\mathfrak{h}$ the Lie algebra of H and by $\mathfrak{g}$ the Lie algebra of G. We shall say that G is a *reductive Lie subgroup* of H if there exists a direct sum decomposition

$$\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{m},$$

where $\mathfrak{m}$ is an Ad_G-invariant vector subspace of $\mathfrak{h}$, i.e. $\mathrm{Ad}_a(\mathfrak{m}) \subset \mathfrak{m}$ for all $a \in G$ (which means that the Ad_G representation of G in $\mathfrak{h}$ is reducible into a direct sum decomposition of two Ad_G-invariant vector spaces; cf. [18, p. 83]).

Remark 3.2. A Lie algebra $\mathfrak{h}$ and a Lie subalgebra $\mathfrak{g}$ satisfying these properties form a so-called *reductive pair* (cf. [4, p. 103]). Moreover, $\mathrm{Ad}_G(\mathfrak{m}) \subset \mathfrak{m}$ implies $[\mathfrak{g}, \mathfrak{m}] \subset \mathfrak{m}$, and, conversely, if G is connected, $[\mathfrak{g}, \mathfrak{m}] \subset \mathfrak{m}$ implies $\mathrm{Ad}_G(\mathfrak{m}) \subset \mathfrak{m}$ [19, p. 190].

Example 3.3. Consider a subgroup $G \subset H$ and suppose that an Ad_G-invariant metric K can be assigned on the Lie algebra $\mathfrak{h}$ (e.g. if H is a semisimple Lie group, K could be the Cartan–Killing form: indeed, this form is Ad_H-invariant and, in particular, also Ad_G-invariant). Set

$$\mathfrak{m} := \mathfrak{g}^{\perp} \equiv \{v \in \mathfrak{h} \mid K(v, u) = 0 \quad \forall u \in \mathfrak{g}\}.$$

Obviously, $\mathfrak{h}$ can be decomposed as the direct sum $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{m}$ and it is easy to show that, under the assumption of Ad_G-invariance of K, the vector subspace $\mathfrak{m}$ is also Ad_G-invariant.

Example 3.4 (The unimodular group). The unimodular group $\mathrm{SL}(m, \mathbb{R})$ is an example of a reductive Lie subgroup of $\mathrm{GL}(m, \mathbb{R})$. To see this, first recall that its Lie algebra $\mathfrak{sl}(m, \mathbb{R})$ is formed by all $m \times m$ traceless matrices. If M is any matrix in $\mathfrak{gl}(m, \mathbb{R})$, the following decomposition holds:

$$\mathsf{M} = \mathsf{U} + \frac{1}{m}\mathrm{tr}(\mathsf{M})\mathsf{I},$$

where $\mathsf{I} := \mathrm{id}_{\mathfrak{gl}(m,\mathbb{R})}$ and U is traceless. Indeed,

$$\mathrm{tr}(\mathsf{U}) = \mathrm{tr}(\mathsf{M}) - \frac{1}{m}\mathrm{tr}(\mathsf{M})\,\mathrm{tr}(\mathsf{I}) = 0.$$

Accordingly, the Lie algebra $\mathfrak{gl}(m, \mathbb{R})$ can be decomposed as follows:

$$\mathfrak{gl}(m, \mathbb{R}) = \mathfrak{sl}(m, \mathbb{R}) \oplus \mathbb{R}\mathsf{I}.$$

In this case, $\mathfrak{m}$ is the set of all real multiples of I, which is obviously adjoint-invariant under $\mathrm{SL}(m, \mathbb{R})$. Indeed, if S is an arbitrary element of $\mathrm{SL}(m, \mathbb{R})$, for any $a \in \mathbb{R}$ one has

$$\mathrm{Ad}_{\mathsf{S}}(a\mathsf{I}) \equiv \mathsf{S}(a\mathsf{I})\mathsf{S}^{-1} = a\mathsf{I}\mathsf{S}\mathsf{S}^{-1} = a\mathsf{I}.$$

This proves that $\mathbb{R}\mathsf{I}$ is adjoint-invariant under $\mathrm{SL}(m, \mathbb{R})$, and $\mathrm{SL}(m, \mathbb{R})$ is a reductive Lie subgroup of $\mathrm{GL}(m, \mathbb{R})$.

Given the importance of the following example for the future developments of the theory, we shall state it as a proposition.

Proposition 3.5. *The* (*pseudo-*) *orthogonal group* $\mathrm{SO}(p, q)$, $p + q = m$, *is a reductive Lie subgroup of* $\mathrm{GL}(m, \mathbb{R})$.

Proof. Let η denote the standard metric of signature (p, q), with $p + q = m$, on $\mathbb{R}^m \equiv \mathbb{R}^{p,q}$ and M be any matrix in $\mathfrak{gl}(m, \mathbb{R})$. Denote by $\mathsf{M}^\top$ the adjoint (transpose) of M with respect to η, defined by requiring $\eta(\mathsf{M}^\top v, v') = \eta(v, \mathsf{M}v')$ for all $v, v' \in \mathbb{R}^m$. Of course, any traceless matrix can be (uniquely) written as the sum of an antisymmetric matrix and a symmetric traceless matrix. Therefore

$$\mathfrak{sl}(m, \mathbb{R}) = \mathfrak{so}(p, q) \oplus \mathbb{V},$$

$\mathfrak{so}(p, q)$ denoting the Lie algebra of the (pseudo-) orthogonal group $SO(p, q)$ for η, formed by all matrices A in $\mathfrak{gl}(m, \mathbb{R})$ such that $\mathsf{A}^\top = -\mathsf{A}$, and $\mathbb{V}$ the vector space of all matrices V in $\mathfrak{sl}(m, \mathbb{R})$ such that $\mathsf{V}^\top = \mathsf{V}$. Now, let O be any element of $SO(p, q)$ and set $\mathsf{V}' := \mathrm{Ad}_{\mathsf{O}} \mathsf{V} \equiv \mathsf{O}\mathsf{V}\mathsf{O}^{-1}$ for any $\mathsf{V} \in \mathbb{V}$. We have

$$\mathsf{V}'^\top = (\mathsf{O}\mathsf{V}\mathsf{O}^\top)^\top = \mathsf{V}'$$

because $\mathsf{V}^\top = \mathsf{V}$ and $\mathsf{O}^{-1} = \mathsf{O}^\top$. Moreover,

$$\mathrm{tr}(\mathsf{V}') = \mathrm{tr}(\mathsf{O})\,\mathrm{tr}(\mathsf{V})\,\mathrm{tr}(\mathsf{O}^{-1}) = 0$$

since V is traceless. So, V' is in $\mathbb{V}$, thereby proving that $\mathbb{V}$ is adjoint-invariant under $SO(p, q)$. Therefore, $SO(p, q)$ is a reductive Lie subgroup of $SL(m, \mathbb{R})$ and, hence, also a reductive Lie subgroup of $GL(m, \mathbb{R})$ by virtue of Example 3.4. $\square$

Definition 3.6. A *reductive G-structure* on a principal bundle $Q(M, H)$ is a principal sub-bundle $P(M, G)$ of $Q(M, H)$ such that G is a reductive Lie subgroup of H.

Now, since later on we shall consider the case of spinor fields, it is convenient to give the following general definition.

Definition 3.7. Let $P(M, G)$ be a principal bundle and $\rho : \Gamma \to G$ a central homomorphism of a Lie group Γ onto G, i.e. such that its kernel is discrete and contained in the centre of Γ [14] (see also [15]). A *Γ-structure* on $P(M, G)$ is a principal bundle map $\zeta : \tilde{P} \to P$ which is equivariant under the right actions of the structure groups, i.e.

$$\zeta(\tilde{u} \cdot \alpha) = \zeta(\tilde{u}) \cdot \rho(\alpha)$$

for all $\tilde{u} \in \tilde{P}$ and $\alpha \in \Gamma$.

Equivalently, we have the following commutative diagrams:

$$
\begin{array}{ccc}
\tilde{P} & \xrightarrow{\;\zeta\;} & P \\
{\scriptstyle\tilde{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\
M & \xrightarrow[\mathrm{id}_M]{} & M
\end{array}
\qquad\qquad
\begin{array}{ccc}
\tilde{P} & \xrightarrow{\;\tilde{r}^{\alpha}\;} & \tilde{P} \\
{\scriptstyle\zeta}\downarrow & & \downarrow{\scriptstyle\zeta} \\
P & \xrightarrow[r^{\alpha}]{} & P
\end{array}
$$

r^a and $\tilde{r}^\alpha$ denoting the right multiplication on P and $\tilde{P}$, respectively (see [8]). This means that, for $\tilde{u} \in \tilde{P}$, both $\tilde{u}$ and $\zeta(\tilde{u})$ lie over the same point, and ζ, restricted to any fibre, is a "copy" of ρ, i.e. it is equivalent to it. The existence condition for a Γ-structure on P can be formulated in terms of Čech cohomology [15,14,23].

Remark 3.8. The bundle map $\zeta : \tilde{P} \to P$ is a covering space since its kernel is discrete.

Recall now that for any principal bundle (P, M, π, G) a (*principal*) *automorphism* of P is a diffeomorphism $\Phi : P \to P$ such that $\Phi(u \cdot a) = \Phi(u) \cdot a$ for every $u \in P, a \in G$. Each Φ induces a unique diffeomorphism $\varphi : M \to M$ such that $\pi \circ \Phi = \varphi \circ \pi$. Accordingly, we shall denote by $\mathrm{Aut}(P)$ the group of all principal automorphisms of P. Assume that a vector field $\varXi$ on P generates a local 1-parameter group $\{\Phi_t\}$. Then, $\varXi$ is *G-invariant* if and only if Φ_t is an automorphism of P for every $t \in \mathbb{R}$. Accordingly, we denote by $\mathfrak{X}_G(P)$ the Lie algebra of G-invariant vector fields on P.

Now recall that, given a fibred manifold $\pi : B \to M$, a *projectable vector field* on B over a vector field ξ on M is a vector field $\varXi$ on B such that $T\pi \circ \varXi = \xi \circ \pi$. Then, the following proposition is evident.

Proposition 3.9. *Let $P(M, G)$ be a principal bundle. Then, every G-invariant vector field $\varXi$ on P is projectable over a unique vector field ξ on the base manifold M.*

Proposition 3.10. *Let $\zeta : \tilde{P} \to P$ be a Γ-structure on $P(M, G)$. Then, every G-invariant vector field $\varXi$ on P admits a unique (Γ-invariant) lift $\tilde{\varXi}$ onto $\tilde{P}$.*

Proof. Consider a G-invariant vector field $\varXi$, its flow being denoted by $\{\Phi_t\}$. For each $t \in \mathbb{R}$, Φ_t is an automorphism of P. Moreover, $\zeta : \tilde{P} \to P$ being a covering space, it is possible to lift Φ_t to a (unique) bundle map $\tilde{\Phi}_t : \tilde{P} \to \tilde{P}$ in the following way. For any point $\tilde{u} \in \tilde{P}$, consider the (unique) point $\zeta(\tilde{u}) = u$. From the theory of covering spaces it follows that, for the curve $\gamma_u : \mathbb{R} \to P$ based at u, that is $\gamma_u(0) = u$, and defined by $\gamma_u(t) := \Phi_t(u)$, there exists a unique curve $\tilde{\gamma}_{\tilde{u}} : \mathbb{R} \to \tilde{P}$ based at $\tilde{u}$ such that $\zeta \circ \tilde{\gamma}_{\tilde{u}} = \gamma_u$. It is possible to define a principal bundle map $\tilde{\Phi}_t : \tilde{P} \to \tilde{P}$ covering Φ_t by setting $\tilde{\Phi}_t(\tilde{u}) := \tilde{\gamma}_{\tilde{u}}(t)$. The 1-parameter group of automorphisms $\{\tilde{\Phi}_t\}$ of $\tilde{P}$ defines a vector field $\tilde{\varXi}(\tilde{u}) := \partial/\partial t[\tilde{\Phi}_t(\tilde{u})]|_{t=0}$ for all $\tilde{u} \in \tilde{P}$. $\qquad\square$

Proposition 3.11. *Let $\zeta : \tilde{P} \to P$ be a Γ-structure on $P(M, G)$. Then, every G-invariant vector field $\tilde{\varXi}$ on $\tilde{P}$ is projectable over a unique G-invariant vector field $\varXi$ on P.*

Proof. Consider a Γ-invariant vector field $\tilde{\varXi}$ on $\tilde{P}$. Denote its flow by $\{\tilde{\Phi}_t\}$. Each $\tilde{\Phi}_t$ induces a unique automorphism $\Phi_t : P \to P$ such that $\zeta \circ \tilde{\Phi}_t = \Phi_t \circ \zeta$ and, hence, a unique vector field $\varXi$ on P given by $\varXi(u) := \partial/\partial t[\Phi_t(u)]|_{t=0}$ for all $u \in P$. $\qquad\square$

Corollary 3.12. *Let $\zeta : \tilde{P} \to P$ be a Γ-structure on $P(M, G)$. There is a bijection between G-invariant vector fields on P and Γ-invariant vector fields on $\tilde{P}$.*

4. Gauge-natural bundles

In this section we shall introduce the category of gauge-natural bundles [6,20] and give a number of relevant examples. Geometrically, gauge-natural bundles possess a very rich structure, which generalizes the classical one of natural bundles. From the physical point

of view, this framework enables one to treat at the same time, under a unifying formalism, natural field theories such as general relativity, gauge theories, as well as bosonic and fermionic matter field theories (cf. [8,9,12,25]).

Definition 4.1. Let $j_p^\ell f$ denote the ℓth order jet prolongation of a map f evaluated at a point p. The set

$$\{j_0^k \alpha | \alpha : \mathbb{R}^m \to \mathbb{R}^m, \alpha(0) = 0, \text{ locally invertible}\}$$

equipped with the jet composition $j_0^k \alpha \circ j_0^k \alpha' := j_0^k(\alpha \circ \alpha')$ is a Lie group called the *kth differential group* and denoted by G_m^k.

For $k = 1$ we have, of course, the identification $G_m^1 \cong \text{GL}(m, \mathbb{R})$.

Definition 4.2. Let M be an m-dimensional manifold. The principal bundle over M with group G_m^k is called the *kth order frame bundle* over M and will be denoted by $L^k M$.

For $k = 1$ we have, of course, the identification $L^1 M \cong LM$, where LM is the usual (principal) *bundle of linear frames* over M (cf. e.g. [18]).

Definition 4.3. Let G be a Lie group. Then, the *space of (m, h)-velocities* of G is defined as

$$T_m^h G := \{j_0^h a | a : \mathbb{R}^m \to G\}.$$

Thus, $T_m^h G$ denotes the set of h-jets with source at the origin $0 \in \mathbb{R}^m$ and target in G. It is a subset of the manifold $J^h(\mathbb{R}^m, G)$ of r-jets with source in $\mathbb{R}^m$ and target in G. The set $J^h(\mathbb{R}^m, G)$ is a fibre bundle over $\mathbb{R}^m$ with respect to the canonical jet projection of $J^h(\mathbb{R}^m, G)$ on $\mathbb{R}^m$, and $T_m^h G$ is its fibre over $0 \in \mathbb{R}^m$. Moreover, the set $T_m^h G$ can be given the structure of a Lie group. Indeed, let $S, T \in T_m^h G$ be any elements. We define a (smooth) multiplication in $T_m^h G$ by

$$T_m^h \mu : T_m^h G \times T_m^h G \to T_m^h G, \quad T_m^h \mu : (S = j_0^h a, T = j_0^h b) \mapsto S \cdot T := j_0^h(a \cdot b),$$

where $(a \cdot b)(x) := a(x) \cdot b(x) \equiv \mu(a(x), b(x))$ is the group multiplication in G. The mapping $(S, T) \mapsto S \cdot T$ is associative; moreover, the element $j_0^h e$, e denoting both the unit element in G and the constant mapping from $\mathbb{R}^m$ to e, is the unit element of $T_m^h G$, and $j_0^r a^{-1}$, where $a^{-1}(x) := (a(x))^{-1}$ (the inversion being taken in the group G), is the inverse of $j_0^h a$.

Definition 4.4. Consider a principal bundle $P(M, G)$. Let k and h be two natural numbers such that $k \geq h$. Then, by the *(k, h)-principal prolongation* of P we shall mean the bundle

$$W^{k,h} P := L^k M \times_M J^h P, \tag{4.1}$$

where $L^k M$ is the kth order frame bundle of M and $J^h P$ the hth order jet prolongation of P. A point of $W^{k,h} P$ is of the form $(j_0^k \epsilon, j_x^h \sigma)$, where $\epsilon : \mathbb{R}^m \to M$ is locally invertible and such that $\epsilon(0) = x$, and $\sigma : M \to P$ is a local section around the point $x \in M$.

Unlike $J^h P$, $W^{k,h} P$ is a principal bundle over M whose structure group is

$$W_m^{k,h} G := G_m^k \rtimes T_m^h G.$$

$W_m^{k,h} G$ is called the $(m; k, h)$-*principal prolongation* of G. The group multiplication on $W_m^{k,h} G$ is defined by the following rule:

$$(j_0^k \alpha, j_0^h a) \odot (j_0^k \beta, j_0^h b) := (j_0^k (\alpha \circ \beta), j_0^h ((a \circ \beta) \cdot b)),$$

'·' denoting the group multiplication in G. The right action of $W_m^{k,h} G$ on $W^{k,h} P$ is then defined by

$$(j_0^k \epsilon, j_x^h \sigma) \odot (j_0^k \alpha, j_0^h a) := (j_0^k (\epsilon \circ \alpha), j_x^h (\sigma \cdot (a \circ \alpha^{-1} \circ \epsilon^{-1}))),$$

'·' denoting now the canonical right action of G on P.

Definition 4.5. Let $\Phi : P \to P$ be an automorphism over a diffeomorphism $\varphi : M \to M$. We define an *automorphism of $W^{k,h} P$ associated with Φ* by

$$W^{k,h} \Phi : W^{k,h} P \to W^{k,h} P, \quad W^{k,h} \Phi : (j_0^k \epsilon, j_x^h \sigma) \mapsto (j_0^k (\varphi \circ \epsilon), j_x^h (\Phi \circ \sigma \circ \varphi^{-1})). \tag{4.2}$$

Proposition 4.6. *The bundle morphism $W^{k,h} \Phi$ preserves the right action, thereby being a principal automorphism.*

By virtue of (4.1) and (4.2) $W^{k,h}$ turns out to be a functor from the category of principal G-bundles over m-dimensional manifolds and local isomorphisms to the category of principal $W_m^{k,h} G$-bundles [20]. Now, let $P_\lambda := W^{k,h} P \times_\lambda F$ be a fibre bundle associated with $P(M, G)$ via an action λ of $W_m^{k,h} G$ on a manifold F. There exists canonical representation of the automorphisms of P induced by (4.2). Indeed, if $\Phi : P \to P$ is an automorphism over a diffeomorphism $\varphi : M \to M$, then we can define the corresponding *induced automorphism* Φ_λ as

$$\Phi_\lambda : P_\lambda \to P_\lambda, \qquad \Phi_\lambda : [u, f]_\lambda \mapsto [W^{k,h} \Phi(u), f]_\lambda, \tag{4.3}$$

which is well defined since it turns out to be independent of the representative $(u, f)_\lambda$, $u \in P$, $f \in F$. This construction yields a functor $\cdot_\lambda$ from the category of principal G-bundles to the category of fibred manifolds and fibre-respecting mappings.

Definition 4.7. A *gauge-natural bundle of order* (k, h) over M associated with $P(M, G)$ is any such functor.

If we now restrict attention to the case $G = \{e\}$ and $h = 0$, we can recover the classical notion of natural bundles over M. In particular, we have the following definition.

Definition 4.8. Let $\varphi : M \to M$ be a diffeomorphism. We define an automorphism of $L^k M$ associated with φ, called its *natural lift*, by

$$L^k \varphi : L^k M \to L^k M, \qquad L^k \varphi : j_0^k \epsilon \mapsto j_0^k (\varphi \circ \epsilon).$$

Then, L^k turns out to be a functor from the category of m-dimensional manifolds and local diffeomorphisms to the category of principal G_m^k-bundles. Now, given any fibre bundle associated with $L^k M$ and any diffeomorphism on M, we can define a corresponding induced automorphism in the usual fashion. This construction yields a functor from the category of m-dimensional manifolds to the category of fibred manifolds.

Definition 4.9. A *natural bundle of order k* over M is any such functor.

We shall now give some important examples of (gauge-) natural bundles.

Example 4.10 (Bundle of tensor densities). A first fundamental example of a natural bundle is given, of course, by the bundle ${}^w T_s^r M$ of tensor densities of weight w over an m-dimensional manifold M. Indeed, ${}^w T_s^r M$ is a vector bundle associated with $L^1 M$ via the following left action of $G_m^1 \cong W_m^{1,0}\{e\}$ on the vector space $T_s^r(\mathbb{R}^m)$:

$$\lambda : G_m^1 \times T_s^r(\mathbb{R}^m) \to T_s^r(\mathbb{R}^m),$$
$$\lambda : (\alpha_k^j, t_{q_1\ldots q_s}^{p_1\ldots p_r}) \mapsto \alpha_{k_1}^{p_1} \cdots \alpha_{k_r}^{p_r} t_{l_1\ldots l_s}^{k_1\ldots k_r} \tilde{\alpha}_{q_1}^{l_1} \cdots \tilde{\alpha}_{q_s}^{l_s} (\det \alpha)^{-w},$$

the tilde over a symbol denoting matrix inversion. For $w = 0$ we recover the bundle of tensor fields over M. This is a definition of ${}^w T_s^r M$ which is appropriate for physical applications, where one usually considers *only* those (active) transformations of tensor fields that are *naturally* induced by some transformations on the base manifold. Somewhat more unconventionally, though, we can regard ${}^w T_s^r M$ as a *gauge*-natural vector bundle associated with $W^{0,0}(LM)$. Of course, the two bundles under consideration are the same *as objects*, but their *morphisms* are different.

Example 4.11 (Bundle of G-invariant vector fields). Let $\mathcal{V} := \mathbb{R}^m \oplus \mathfrak{g}$, $\mathfrak{g}$ denoting the Lie algebra of G, and consider the following action:

$$\lambda : W_m^{1,1} G \times \mathcal{V} \to \mathcal{V}, \quad \lambda : ((\alpha_k^j, a^q, a_l^r), (v^i, v^p)) \mapsto (\alpha_j^i v^j, A_q^p(a)(v^q + a_j^q v^j)), \tag{4.4}$$

where (a^q, a_l^r) denote natural coordinates on $T_m^1 G$: a generic element $j_0^1 f \in T_m^1 G$ is represented by $a = f(0) \in G$, i.e. $a^q = f^q(0)$, and $a_l^r = (\partial_l(a^{-1} f(x))|_{x=0}^r$. Obviously, $W^{1,1} P \times_\lambda \mathcal{V} \cong TP/G$, its sections thus representing G-invariant vector fields on P.

Example 4.12 (Bundle of vertical G-invariant vector fields). Take $\mathfrak{g}$ as the standard fibre and consider the following action:

$$\lambda : W_m^{1,1} G \times \mathfrak{g} \to \mathfrak{g}, \qquad \lambda : ((\alpha_k^j, a^q, a_l^r), v^p) \mapsto A_q^p(a)v^q. \tag{4.5}$$

It is easy to realize that $W^{1,1} P \times_\lambda \mathfrak{g} \cong VP/G \cong (P \times \mathfrak{g})/G$, the bundle of vertical G-invariant vector fields on P. Of course, in this example, we see that $\mathfrak{g}$ is already a G-manifold and so $(P \times \mathfrak{g})/G$ is a gauge-natural bundle of order $(0, 0)$, i.e. a vector bundle associated with $W^{0,0} P \cong P$. In other words, giving action (4.5) amounts to regarding the original G-manifold $\mathfrak{g}$ as a $W_m^{1,1} G$-manifold via the canonical projection of Lie groups

$W_m^{1,1}G \to G$. It is also meaningful to think of action (4.5) as setting $v^i = 0$ in (4.4), and hence one sees that the first jet contribution, i.e. a_i^p, disappears.

5. Split structures on principal bundles

It is known that, given a principal bundle $P(M, G)$, a *principal connection* on P may be viewed as a fibre G-equivariant projection $\Phi : TP \to VP$, i.e. as a 1-form in $\Omega^1(P, TP)$ such that $\Phi \circ \Phi = \Phi$ and im $\Phi = VP$. Here, "G-equivariant" means that $Tr^a \circ \Phi = \Phi \circ Tr^a$ for all $a \in G$. Then, $HP := \ker \Phi$ is a constant-rank vector subbundle of TP, called the *horizontal bundle*. We have a decomposition $TP = HP \oplus VP$ and $T_u P = H_u P \oplus V_u P$ for all $u \in P$. The projection Φ is called the *vertical projection* and the projection $\chi := \mathrm{id}_{TP} - \Phi$, which is also G-equivariant and satisfies $\chi \circ \chi = \chi$ and im $\chi = \ker \Phi$, is called the *horizontal projection*.

This is, of course, a well-known example of a "split structure" on a principal bundle. We shall now give the following general definition, due—for pseudo-Riemannian manifolds— to a number of authors [3,10,13,28,29] and more generally to Gladush and Konoplya [11].

Definition 5.1. An *r-split structure* on a principal bundle $P(M, G)$ is a system of r fibre G-equivariant linear operators $\{\Phi^i \in \Omega^1(P, TP)\}$, $i = 1, 2, \ldots, r$, of constant rank with the properties:

$$\Phi^i \cdot \Phi^j = \delta^{ij} \Phi^j, \qquad \sum_{i=1}^{r} \Phi^i = \mathrm{id}_{TP}. \qquad (5.1)$$

We introduce the notations:

$$\Sigma_u^i := \mathrm{im}\, \Phi_u^i, \qquad n_i := \dim \Sigma_u^i, \qquad (5.2)$$

where im Φ_u^i is the image of the operator Φ^i at a point u of P, i.e. $\Sigma_u^i = \{v \in T_u P | \Phi_u^i \circ v = v\}$. Owing to the constancy of the rank of the operators $\{\Phi^i\}$, the numbers $\{n_i\}$ do not depend on the point u of P. It follows from the very definition of an r-split structure that we have a G-equivariant decomposition of the tangent space:

$$T_u P = \bigoplus_{i=1}^{r} \Sigma_u^i, \qquad \dim T_u P = \sum_{i=1}^{r} n_i.$$

Obviously, the bundle TP is also decomposed into r vector subbundles $\{\Sigma^i\}$ so that

$$TP = \bigoplus_{i=1}^{r} \Sigma^i, \qquad \Sigma^i = \bigcup_{u \in P} \Sigma_u^i. \qquad (5.3)$$

Remark 5.2. In general, the r vector subbundles $\{\Sigma^i \to P\}$ are *anholonomic*, i.e. non-integrable, and are not vector subbundles of VP. For a principal connection, i.e. for the case $TP = HP \oplus VP$, the subbundle VP is integrable.

Proposition 5.3. *An equivariant decomposition of TP into r vector subbundles $\{\Sigma^i\}$ as given by* (4.3), *with* $T_u r^a(\Sigma^i_u) = \Sigma^i_{u \cdot a}$, *induces a system of r fibre G-equivariant linear operators* $\{\Phi^i \in \Omega^1(P, TP)\}$ *of constant-rank satisfying properties* (5.1) *and* (5.2).

Proposition 5.4. *Given an r-split structure on a principal bundle $P(M, G)$, every G-invariant vector field Ξ on P splits into r invariant vector fields $\{\Xi_i\}$ such that $\Xi = \Xi_1 \oplus \cdots \oplus \Xi_r$ and $\Xi_i(u) \in \Sigma^i$ for all $u \in P$ and $i = 1, 2, \ldots, r$.*

Remark 5.5. The vector fields $\{\Xi_i\}$ are compatible with the $\{\Sigma^i\}$, i.e. they are sections $\{\Xi_i : P \to \Sigma^i\}$ of the vector bundles $\{\Sigma^i \to P\}$.

Corollary 5.6. *Let $P(M, G)$ be a reductive G-structure on a principal bundle $Q(M, H)$ and let $i_P : P \to Q$ be the canonical embedding. Then, any given r-split structure on $Q(M, H)$ induces an r-split structure restricted to $P(M, G)$, i.e. an equivariant decomposition of $i_P^*(TQ) \equiv P \times_Q TQ = \{(u, v) \in P \times TQ | i_P(u) = \tau_Q(v)\}$ such that $i_P^*(TQ) = i_P^*(\Sigma^1) \oplus \cdots \oplus i_P^*(\Sigma^r)$, and any H-invariant vector field Ξ on Q restricted to P splits into r G-invariant sections of the pull-back bundles $\{i_P^*(\Sigma^i) \equiv P \times_Q \Sigma^i\}$, i.e. $\Xi = \Xi_1 \oplus \cdots \oplus \Xi_r$ with $\Xi_i(u) \in \Sigma^i$ for all $u \in P$ and $i \in \{1, 2, \ldots, r\}$.*

Remark 5.7. Note that the pull-back i_P^* is a *natural operation*, i.e. it respects the splitting $i_P^*(TQ) = i_P^*(\Sigma^1) \oplus \cdots \oplus i_P^*(\Sigma^r)$. In other words, the pull-back of a splitting for Q is a splitting of the pull-backs for P. Furthermore, although the vector fields $\{\Xi_i\}$ are G-invariant sections of their respective pull-back bundles, they are H-invariant if regarded as vector fields on the corresponding subsets of Q.

In Section 4 we saw that $W^{k,h}P$ is a principal bundle over M. Consider in particular $W^{1,1}P$, the $(1, 1)$-principal prolongation of P. The fibred manifold $W^{1,1}P \to M$ coincides with the fibred product $W^{1,1}P := L^1M \times_M J^1P$ over M. We have two canonical principal bundle morphisms $\mathrm{pr}_1 : W^{1,1}P \to L^1M$ and $\mathrm{pr}_2 : W^{1,1}P \to P$. In particular, $\mathrm{pr}_2 : W^{1,1}P \to P$ is a $G^1_m \rtimes \mathfrak{g} \otimes \mathbb{R}^m$-principal bundle, $G^1_m \rtimes \mathfrak{g} \otimes \mathbb{R}^m$ being the kernel of $W^1_m G \to G$. The following lemma recognizes $\tau_P : TP \to P$ as a vector bundle associated with the principal bundle $W^{1,1}P \to P$.

Lemma 5.8. *The vector bundle $\tau_P : TP \to P$ coincides with the vector bundle $T^{1,1}P := (W^{1,1}P \times \mathcal{V})/(G^1_m \rtimes \mathfrak{g} \otimes \mathbb{R}^m)$ over P, where $\mathcal{V} := \mathbb{R}^m \oplus \mathfrak{g}$ is the left $G^1_m \rtimes \mathfrak{g} \otimes \mathbb{R}^m$-manifold with action given by*

$$\tau : G^1_m \rtimes \mathfrak{g} \otimes \mathbb{R}^m \times \mathcal{V} \to \mathcal{V}, \qquad \tau : ((\alpha^j_k, e, a^r_l), (v^i, v^p)) \mapsto (\alpha^i_j v^j, v^p + a^p_i v^i).$$

$$(5.4)$$

Remark 5.9. The vector bundle $\tau_P : TP \to P$ is a gauge-natural bundle of order $(0, 0)$ associated with the $G^1_m \rtimes \mathfrak{g} \otimes \mathbb{R}^m$-principal bundle $\mathrm{pr}_2 : W^{1,1}P \to P$.

Lemma 5.10. *$VP \to P$ is a trivial vector bundle associated with $W^{1,1}P \to P$.*

Lemma 5.11. *Let $P(M, G)$ be a reductive G-structure on a principal bundle $Q(M, H)$ and $i_P : P \to Q$ the canonical embedding. Then, $i_P^*(TQ) = P \times_Q TQ$ is a vector bundle over P associated with $W^{1,1}P \to P$.*

From the above lemmas it follows that another important example of a split structure on a principal bundle is given by the following theorem.

Theorem 5.12. *Let $P(M, G)$ be a reductive G-structure on a principal bundle $Q(M, H)$ and let $i_P : P \to Q$ be the canonical embedding. Then, there exists a canonical decomposition of $i_P^*(TQ) \to P$ such that*

$$i_P^*(TQ) = TP \oplus \mathcal{M}(P),$$

i.e. at each $u \in P$ one has

$$T_u Q = T_u P \oplus \mathcal{M}_u,$$

$\mathcal{M}_u$ being the fibre over u of the subbundle $\mathcal{M}(P) \to P$ of $i_P^*(VQ) \to P$. The bundle $\mathcal{M}(P)$ is defined as $\mathcal{M}(P) := (W^{1,1}P \times \mathfrak{m})/(G_m^1 \rtimes \mathfrak{g} \otimes \mathbb{R}^m)$, where $\mathfrak{m}$ is the (trivial left) $G_m^1 \rtimes \mathfrak{g} \otimes \mathbb{R}^m$-manifold.

Remark 5.13. The trivial $G_m^1 \rtimes \mathfrak{g} \otimes \mathbb{R}^m$-manifold $\mathfrak{m}$ corresponds to the action (4.5) of Example 4.12 with $W_m^{1,1}G$ restricted to $G_m^1 \rtimes \mathfrak{g} \otimes \mathbb{R}^m$, and $\mathfrak{g}$ restricted to $\mathfrak{m}$. Of course, since the group $G_m^1 \rtimes \mathfrak{g} \otimes \mathbb{R}^m$ acts trivially on $\mathfrak{m}$, it follows that $\mathcal{M}(P)$ is trivial, i.e. isomorphic to $P \times \mathfrak{m}$, because $W^{1,1}P/(G_m^1 \rtimes \mathfrak{g} \otimes \mathbb{R}^m) \cong P$.

From the above theorem two corollaries follow, which are of prime importance for the concepts of a Lie derivative we shall introduce in the next section.

Corollary 5.14. *Let $P(M, G)$ and $Q(M, H)$ be as in the previous theorem. The restriction $\Xi|_P$ of an H-invariant vector field Ξ on Q to P splits into a G-invariant vector field Ξ_K on P, called the* generalized Kosmann vector field *associated with Ξ, and a "transverse" vector field Ξ_G, called the* generalized von Göden vector field *associated with Ξ.*

Corollary 5.15. *Let $P(M, G)$ be a classical G-structure, i.e. a reductive G-structure on the bundle LM of linear frames over M. The restriction $L\xi|_P$ to $P \to M$ of the natural lift $L\xi$ onto LM of a vector field ξ on M splits into a G-invariant vector field on P called the* generalized Kosmann lift *of ξ and denoted simply by ξ_K, and a "transverse" vector field called the* von Göden lift *of ξ and denoted by ξ_G.*

Remark 5.16. The last corollary still holds if, instead of LM, one considers the kth order frame bundle $L^k M$ and hence a classical G-structure of order k, i.e. a reductive G-subbundle P of $L^k M$. Note also that the Kosmann lift $\xi \mapsto \xi_K$ is *not* a Lie algebra homomorphism, although ξ_K is a G-invariant vector field and projects over ξ.

Example 5.17 (Kosmann lift). A fundamental example of a G-structure on a manifold M is given, of course, by the bundle SO(M, g) of its (pseudo-) orthonormal frames with respect

to a metric g of signature (p, q), where $p + q = m \equiv \dim M$. SO(M, g) is a principal bundle (over M) with structure group $G = $ SO(p, q). Now, recall that the natural lift of a vector field ξ onto LM is defined as

$$L\xi := \left. \frac{\partial}{\partial t} L^1 \varphi_t \right|_{t=0},$$

$\{\varphi_t\}$ denoting the flow of ξ. If (ρ_a^b) denotes a (local) basis of right GL$(m, \mathbb{R})$-invariant vector fields on LM reading $(\rho_a^b = u_c^b \partial/\partial u_c^a)$ in some local chart (x^μ, u_a^b) and $(e_a =: e_a^\mu \partial_\mu)$ is a local section of LM, then $L\xi$ has the local expression

$$L\xi = \xi^a e_a + (L\xi)_b^a \rho_a^b,$$

where $\xi =: \xi^a e_a$ and

$$(L\xi)_b^a := \tilde{e}_\rho^a (\partial_\nu \xi^\rho e_b^\nu - \xi^\nu \partial_\nu e_b^\rho).$$

If we now let (e_a) and (x^μ, u_a^b) denote a local section and a local chart of SO(M, g), respectively, then the generalized Kosmann lift $\xi_{\rm K}$ on SO(M, g) of a vector field ξ on M, simply called its *Kosmann lift* [7], locally reads

$$\xi_{\rm K} = \xi^a e_a + (L\xi)_{[ab]} A^{ab},$$

where (A^{ab}) is a basis of right SO(q, p)-invariant vector fields on SO(M, g) locally reading $(A^{ab} = \eta^{c[a} \delta_d^{b]} \rho_c^d)$, $(L\xi)_{ab} := \eta_{ac}(L\xi)_b^c$, and (η_{ac}) denote the components of the standard Minkowski metric of signature (p, q).

Now, combining Proposition 3.10 and Theorem 5.12 yields the following result, which, in particular, will enable us to extend the concept of a Kosmann lift to the important context of spinor fields.

Corollary 5.18. *Let $\zeta : \tilde{P} \to P$ be a Γ-structure over a classical G-structure $P(M, G)$. Then, the generalized Kosmann lift $\xi_{\rm K}$ of a vector field ξ on M lifts to a unique $(\Gamma$-invariant$)$ vector field $\tilde{\xi}_{\rm K}$ on $\tilde{P}$, which projects over $\xi_{\rm K}$.*

6. Lie derivatives on reductive *G*-structures

As already mentioned in Section 1, the general theory of Lie derivatives stems from Trautman's seminal paper [27]. Here, we mainly follow the notation and conventions of [20, Section 47].

Definition 6.1. Let M and N be two manifolds and $f : M \to N$ a map between them. By a *vector field along f* we shall mean a map $Z : M \to TN$ such that $\tau_N \circ Z = f$, $\tau_N : TN \to N$ denoting the canonical tangent bundle projection.

Definition 6.2. Let M, N and f be as above, and let X and Y be two vector fields on M and N, respectively. Then, by the *generalized Lie derivative* $\tilde{\pounds}_{(X,Y)}f$ of f with respect to X and Y we shall mean the vector field along f given by

$$\tilde{\pounds}_{(X,Y)}f := Tf \circ X - Y \circ f.$$

If $\{\varphi_t\}$ and $\{\Phi_t\}$ denote the flows of X and Y, respectively, then one readily verifies that

$$\tilde{\pounds}_{(X,Y)}f = \left. \frac{\partial}{\partial t}(\Phi_{-t} \circ f \circ \varphi_t) \right|_{t=0}.$$

An important specialization of Definition 6.2 is the following

Definition 6.3. Let $\pi : B \to M$ be a fibred manifold, $\sigma : M \to B$ a section of π, and Ξ a projectable vector field on B over a vector field ξ on M. Then, by the *generalized Lie derivative* $\tilde{\pounds}_{\Xi}\sigma$ *of* σ *with respect to* Ξ we shall mean the map

$$\tilde{\pounds}_{\Xi}\sigma := \tilde{\pounds}_{(\xi,\Xi)}\sigma : M \to VB. \tag{6.1}$$

(It is easy to realize that $\tilde{\pounds}_{\Xi}\sigma \equiv T\sigma \circ \xi - \Xi \circ \sigma$ takes indeed values in the vertical tangent bundle simply by applying $T\pi$ to it and remembering that Ξ is projectable.)

Now recall that a fibred manifold $\pi : B \to M$ admits a *vertical splitting* if there exists a linear bundle isomorphism (covering the identity of B) $\alpha : VB \to B \times_M \bar{B}$, where $\bar{\pi} : \bar{B} \to M$ is a vector bundle. In particular, a vector bundle $\pi : B \to M$ admits a *canonical vertical splitting* $\alpha : VB \to B \times_M B$. Indeed, if $\tau_B : TB \to B$ denotes the (canonical) tangent bundle projection restricted to VB, y is a point in B such that $y = \tau_B(v)$ for a given $v \in VB$, and $\gamma : \mathbb{R} \to B_y \equiv \pi^{-1}(\pi(y))$ is a curve such that $\gamma(0) = y$ and $j_0^1\gamma = v$, then α is given by $\alpha(v) := (y, w)$, where $w := \lim_{t \to 0}(\gamma(t) - \gamma(0))/t$.

Proposition 6.4. *In this case, the generalized Lie derivative* $\tilde{\pounds}_{\Xi}\sigma$ *is of the form*

$$\tilde{\pounds}_{\Xi}\sigma = (\sigma, \pounds_{\Xi}\sigma), \tag{6.2}$$

the first component being the original section σ. *The second component* $\pounds_{\Xi}\sigma$ *is a section of* $\bar{B}$, *called the* Lie derivative of σ with respect to Ξ. *For the sake of clarity, the operator* $\pounds$ *will be occasionally referred to as the restricted Lie derivative* [20, Section 47].

Remark 6.5. In this case, on using the fact that $\pounds_{\Xi}\sigma$ is the derivative of $\Phi_{-t} \circ \sigma \circ \varphi_t$ at $t = 0$ in the classical sense, one can re-express the restricted Lie derivative in the form

$$\pounds_{\Xi}\sigma(x) = \lim_{t \to 0} \frac{1}{t}(\Phi_{-t} \circ \sigma \circ \varphi_t(x) - \sigma(x)). \tag{6.3}$$

Proposition 6.4 also works whenever B is an *affine* bundle. This is so because, also in this case, $\pi : B \to M$ admits a canonical vertical decomposition $\alpha : VB \to B \times_M \bar{B}$, where $\bar{\pi} : \bar{B} \to M$ is the vector bundle associated with B.

Now, we can specialize Definition 6.3 to the case of gauge-natural bundles in a straightforward manner.

Definition 6.6. Let P_λ be a gauge-natural bundle associated with some principal bundle $P(M, G)$, $\varXi$ a G-invariant vector field on P projecting over a vector field ξ on M, and $\sigma : M \to P_\lambda$ a section of P_λ. Then, by the *generalized* (*gauge-natural*) *Lie derivative of σ with respect to $\varXi$* we shall mean the map

$$\tilde{\pounds}_\varXi \sigma : M \to VP_\lambda, \qquad \tilde{\pounds}_\varXi \sigma := T\sigma \circ \xi - \varXi_\lambda \circ \sigma, \qquad (6.4)$$

where $\varXi_\lambda$ is the generator of the 1-parameter group $\{(\Phi_t)_\lambda\}$ of automorphisms of P_λ functorially induced by the flow $\{\Phi_t\}$ of $\varXi$ [cf. (4.3)]. Equivalently,

$$\tilde{\pounds}_\varXi \sigma = \left. \frac{\partial}{\partial t}((\Phi_{-t})_\lambda \circ \sigma \circ \varphi_t) \right|_{t=0}, \qquad (6.4')$$

$\{\varphi_t\}$ denoting the flow of ξ.

As usual, whenever $\pi : P_\lambda \to M$ admits a canonical vertical splitting of VP_λ, we shall write $\pounds_\varXi \sigma : M \to \bar{P}_\lambda := \overline{P_\lambda}$ for the corresponding restricted Lie derivative.

Furthermore, for each Γ-structure $\zeta : \tilde{P} \to P$ on P, we shall simply write $\pounds_\varXi \tilde{\sigma} := \pounds_{\tilde{\varXi}} \tilde{\sigma} : M \to \bar{\tilde{P}}_{\tilde\lambda}$, $\tilde{P}_{\tilde\lambda}$ denoting a gauge-natural bundle associated with $\tilde{P}$ (admitting a canonical vertical splitting) and $\tilde{\sigma} : M \to \tilde{P}_{\tilde\lambda}$ one of its sections, since $\varXi$ admits a unique (Γ-invariant) lift $\tilde{\varXi}$ onto $\tilde{P}$ (cf. Proposition 3.10). We stress that Definition 6.6 is the conceptually natural generalization of the classical notion of a Lie derivative [30], to which it suitably reduces when applied to natural objects and, hence, notably, to tensor fields and tensor densities.

Of course, we can now further specialize to the case of classical G-structures and, in particular, give the following definition.

Definition 6.7. Let P_λ be a gauge-natural bundle associated with some classical G-structure $P(M, G)$, ξ_K the generalized Kosmann lift (on P) of a vector field ξ on M, and $\sigma : M \to P_\lambda$ a section of P_λ. Then, by the *generalized Lie derivative $\tilde{\pounds}_\xi \sigma$ of σ with respect to ξ* we shall mean the map $\tilde{\pounds}_\xi \sigma := \tilde{\pounds}_{\xi_K} \sigma$, where $\tilde{\pounds}_{\xi_K} \sigma$ denotes the generalized Lie derivative of σ with respect to ξ_K in the sense of Definition 6.6.

Consistently, we shall simply write $\pounds_\xi \sigma := \pounds_{\xi_K} \sigma : M \to \bar{P}_\lambda$ for the corresponding restricted Lie derivative, whenever defined, and $\pounds_\xi \tilde{\sigma} := \pounds_{\tilde\xi_K} \tilde{\sigma} : M \to \bar{\tilde{P}}_{\tilde\lambda}$ for the (restricted) Lie derivative of a section σ of a gauge-natural bundle $\tilde{P}_{\tilde\lambda}$ associated with some principal prolongation of a Γ-structure $\zeta : \tilde{P} \to P$ (and admitting a canonical vertical splitting), which makes sense since ξ_K admits a unique (Γ-invariant) lift $\tilde\xi_K$ onto $\tilde{P}$ (cf. Corollary 5.18).

Example 6.8 (Lie derivative of spinor fields. I). In Example 5.17 we mentioned that a fundamental example of a G-structure on a manifold M is given by the bundle $\mathrm{SO}(M, g)$ of its (pseudo-) orthonormal frames. An equally fundamental example of a Γ-structure on $\mathrm{SO}(M, g)$ is given by the corresponding spin bundle $\mathrm{Spin}(M, g)$ with structure group $\Gamma = \mathrm{Spin}(p, q)$. Now, it is obvious that spinor fields can be regarded as sections of a suitable gauge-natural bundle over M. Indeed, if λ is the linear representation of $\mathrm{Spin}(p, q)$ on the vector space $\mathbb{C}^m$ induced by a given choice of γ matrices, then the associated vector bundle $S(M) := \mathrm{Spin}(M, g) \times_\lambda \mathbb{C}^m$ is a gauge-natural bundle of order $(0, 0)$ whose sections

represent spinor fields (or, more precisely, spin-vector fields). Therefore, in spite of what is sometimes believed, a Lie derivative of spinors (in the sense of Definition 6.6) always exists, *no matter what* the vector field ξ on M is. Locally, such a Lie derivative reads

$$\pounds_\Xi \psi = \xi^a e_a \psi + \tfrac{1}{4} \Xi_{ab} \gamma^a \gamma^b \psi$$

for any spinor field ψ, ($\Xi_{ab} = \Xi_{[ab]}$) denoting the components of an SO(p, q)-invariant vector field $\Xi = \xi^a e_a + \Xi_{ab} A^{ab}$ on SO(M, g), $\xi =: \xi^a e_a$, and $e_a \psi$ the Pfaff derivative of ψ along the local section ($e_a =: e_a^\mu \partial_\mu$) of SO($M, g$) induced by some local section of Spin(M, g). This is the most general notion of a (gauge-natural) Lie derivative of spinor fields and the appropriate one for most situations of physical interest (cf. [12,25]): the generality of Ξ might be disturbing, but is the *unavoidable* indication that $S(M)$ is *not* a natural bundle. If we wish nonetheless to remove such a generality, we must *choose* some canonical (*not* natural) lift of ξ onto SO(M, g). The conceptually (*not* mathematically) most "natural" choice is perhaps given by the Kosmann lift (recall Example 5.17 and use Corollary 5.18). The ensuing Lie derivative locally reads

$$\pounds_\xi \psi = \xi^a e_a \psi + \tfrac{1}{4} (L\xi)_{[ab]} \gamma^a \gamma^b \psi. \tag{6.5}$$

Of course, if '∇' denotes the covariant derivative operator associated with the Levi-Civita (or Riemannian) connection with respect to g, the previous expression can be recast into the form

$$\pounds_\xi \psi = \xi^a \nabla_a \psi - \tfrac{1}{4} \nabla_{[a} \xi_{b]} \gamma^a \gamma^b \psi, \tag{6.5'}$$

which reproduces exactly Kosmann's definition [22] (see [7] for further details and a more thorough discussion). We stress that, although *in this case* its local expression would be identical with (6.5), this is *not* the "metric Lie derivative" introduced by Bourguignon and Gauduchon in [2]. To convince oneself of this it is enough to take the Lie derivative of the metric g, which is a section of the *natural* bundle $\bigvee^2 T^*M$, '$\bigvee$' denoting the symmetrized tensor product. Since the (restricted) Lie derivative $\pounds_\xi$ in the sense of Definition 6.7 must reduce to the ordinary one on natural objects, it holds that

$$\pounds_{L\xi} g = \pounds_\xi g.$$

On the other hand, if $\pounds_\xi$ coincided with the operator $\pounds_\xi^g$ defined by Bourguignon and Gauduchon, the right-hand side of the above identity should equal zero [2, Proposition 15], thereby implying that ξ is a Killing vector field, contrary to the fact that ξ is completely arbitrary. Indeed, in order to recover Bourguignon and Gauduchon's definition, another concept of a Lie derivative must be introduced.

We shall start by recalling two classical definitions [17].

Definition 6.9. Let $P(M, G)$ be a (classical) G-structure. Let φ be a diffeomorphism of M onto itself and $L^1\varphi$ its natural lift onto LM. If $L^1\varphi$ maps P onto itself, i.e. if $L^1\varphi(P) \subseteq P$, then φ is called an *automorphism* of the G-structure P.

Definition 6.10. Let $P(M, G)$ be a G-structure. A vector field ξ on M is called an *infinitesimal automorphism* of the G-structure P if it generates a local 1-parameter group of automorphisms of P.

We can now generalize these concepts to the framework of reductive G-structures as follows.

Definition 6.11. Let $P(M, G)$ be a reductive G-structure on a principal bundle $Q(M, H)$ and Φ a principal automorphism of Q. If Φ maps P onto itself, i.e. if $\Phi(P) \subseteq P$, then Φ is called a *generalized automorphism* of the reductive G-structure P.

Of course, each element of $\mathrm{Aut}(P)$, i.e. each principal automorphism of P, is by definition a generalized automorphism of the reductive G-structure P. Analogously, we have the following definition.

Definition 6.12. Let $P(M, G)$ be a reductive G-structure on a principal bundle $Q(M, H)$. An H-invariant vector field Ξ on Q is called a *generalized infinitesimal automorphism* of the reductive G-structure P if it generates a local 1-parameter group of generalized automorphisms of P.

Of course, each element of $\mathfrak{X}_G(P)$, i.e. each G-invariant vector field on P, is by definition a generalized infinitesimal automorphism of the reductive G-structure P.

Now, along the lines of [19, Proposition X.1.1] it is easy to prove the following

Proposition 6.13. *Let $P(M, G)$ be a reductive G-structure on a principal bundle $Q(M, H)$. An H-invariant vector field Ξ on Q is a generalized infinitesimal automorphism of the reductive G-structure P if and only if Ξ is tangent to P at each point of P.*

We then have the following important lemma.

Lemma 6.14. *Let $P(M, G)$ be a reductive G-structure on a principal bundle $Q(M, H)$ and Ξ a generalized infinitesimal automorphism of the reductive G-structure P. Then, the flow $\{\Phi_t\}$ of Ξ, it being H-invariant, induces on each gauge-natural bundle Q_λ associated with Q a 1-parameter group $\{(\Phi_t)_\lambda\}$ of global automorphisms.*

Proof. Since Ξ is by assumption a generalized infinitesimal automorphism, it is by definition an H-invariant vector field on Q. Therefore, its flow $\{\Phi_t\}$ is a 1-parameter group of H-equivariant maps on the subset P of Q. Then, if $Q_\lambda = W^{k,h} Q \times_\lambda F$, we set

$$(\Phi_t)_\lambda([u, f]_\lambda) := [W^{k,h}\Phi_t(u), f]_\lambda,$$

$u \in Q$, $f \in F$, and are back to the situation of formula (4.3). $\square$

Corollary 6.15. *Let $P(M, G)$ and $Q(M, H)$ be as in the previous lemma, and let Ξ be an H-invariant vector field on Q. Then, the flow $\{(\Phi_{\mathrm{K}})_t\}$ of the generalized Kosmann vector*

field Ξ_K associated with Ξ induces on each gauge-natural bundle Q_λ associated with Q a 1-parameter group $\{((\Phi_K)_t)_\lambda\}$ of global automorphisms.

Proof. Recall that, although the generalized Kosmann vector field Ξ_K is a G-invariant vector field on P, it is H-invariant if regarded as a vector field on the corresponding subset of Q (cf. Remark 5.7 and Corollary 5.15). Therefore, its flow $\{(\Phi_K)_t\}$ is a 1-parameter group of H-equivariant automorphisms on the subset P of Q.

We now want to define a 1-parameter group of automorphisms $\{((\Phi_K)_t)_\lambda\}$ of $Q_\lambda = W^{k,h}Q \times_\lambda F$. Let $[u, f]_\lambda \in Q_\lambda$, $u \in Q$ and $f \in F$, and let u_1 be a point in P such that $\pi(u_1) = \pi(u)$, $\pi : Q \to M$ denoting the canonical projection. There exists a unique $a_1 \in H$ such that $u = u_1 \cdot a_1$. Set then

$$((\Phi_K)_t)_\lambda([u, f]_\lambda) := [W^{k,h}(\Phi_K)_t(u_1), a_1 f]_\lambda.$$

We must show that, given another point $u_2 \in P$ such that $u = u_2 \cdot a_2$ for some (unique) $a_2 \in H$, we have

$$[W^{k,h}(\Phi_K)_t(u_1), a_1 f]_\lambda = [W^{k,h}(\Phi_K)_t(u_2), a_2 f]_\lambda.$$

Indeed, since the action of H is free and transitive on the fibres, from $u = u_1 \cdot a_1$ and $u = u_2 \cdot a_2$ it follows that $a_1 = a \cdot a_2$ or $a = a_1 \cdot (a_2)^{-1}$ or $a_2 = a^{-1} \cdot a_1$. But then

$$
\begin{aligned}
[W^{k,h}(\Phi_K)_t(u_2), a_2 f]_\lambda &= [W^{k,h}(\Phi_K)_t(u_1 \cdot a), a^{-1} \cdot a_1 f]_\lambda \\
&= [W^{k,h}(\Phi_K)_t(u_1) \odot W_m^{k,h}a, a^{-1} \cdot a_1 f]_\lambda \\
&= [W^{k,h}(\Phi_K)_t(u_1), a_1 f]_\lambda
\end{aligned}
$$

as claimed. It is then easy to see that the so-defined $((\Phi_K)_t)_\lambda$ does not depend on the chosen representative. $\qquad\square$

By virtue of the previous corollary, we can now give the following definition.

Definition 6.16. Let $P(M, G)$ be a reductive G-structure on a principal bundle $Q(M, H)$, $G \neq \{e\}$, and Ξ an H-invariant vector field on Q projecting over a vector field ξ on M. Let Q_λ be a gauge-natural bundle associated with Q and $\sigma : M \to Q_\lambda$ a section of Q_λ. Then, by the *generalized G-reductive Lie derivative of σ with respect to Ξ* we shall mean the map

$$\tilde{\pounds}_\Xi^G \sigma := \frac{\partial}{\partial t}(((\Phi_K)_{-t})_\lambda \circ \sigma \circ \varphi_t)\Big|_{t=0},$$

$\{\varphi_t\}$ denoting the flow of ξ.

The corresponding notions of a restricted Lie derivative and a (generalized or restricted) Lie derivative on an associated Γ-structure can be defined in the usual way.

Remark 6.17. When $Q = P$ (and $H = G$), Ξ_K is just Ξ, and we recover the notion of a (generalized) Lie derivative in the sense of Definition 6.6, *but*, as G is required not to equal the trivial group $\{e\}$, Q_λ is never allowed to be a (purely) natural bundle.

By its very definition, the (restricted) *G*-reductive Lie derivative does *not* reduce, in general, to the ordinary (natural) Lie derivative on fibre bundles associated with $L^k M$. This fact makes it unsuitable in all those situations where one needs a *unique* operator which reproduce "standard results" when applied to "standard objects".

In other words, $£_\Xi^G$ is defined with respect to some *pre-assigned* (generalized) symmetries. We shall make this statement explicit in Proposition 6.19 below, which provides a generalization of a well-known classical result.

Let then K be a tensor over the vector space $\mathbb{R}^m$ (i.e. an element of the tensor algebra over $\mathbb{R}^m$) and *G* the group of linear transformations of $\mathbb{R}^m$ leaving K invariant. Recall that each reduction of the structure group $\mathrm{GL}(m, \mathbb{R})$ to *G* gives rise to a tensor field *K* on *M*. Indeed, we may regard each $u \in LM$ as a linear isomorphism of $\mathbb{R}^m$ onto $T_x M$, where $x = \pi(u)$ and $\pi : LM \to M$ denotes, as usual, the canonical projection. Now, if $P(M, G)$ is a *G*-structure, at each point *x* of *M* we can choose a frame *u* belonging to *P* such that $\pi(u) = x$. Since *u* is a linear isomorphism of $\mathbb{R}^m$ onto the tangent space $T_x M$, it induces an isomorphism of the tensor algebra over $\mathbb{R}^m$ onto the tensor algebra over $T_x M$. Then K_x is the image of K under this isomorphism. The invariance of K by *G* implies that K_x is defined independent of the choice of *u* in $\pi^{-1}(x)$. Then, we have the following classical result [17].

Proposition 6.18. *Let* K *be a tensor over the vector space* $\mathbb{R}^m$ *and G the group of linear transformations of* $\mathbb{R}^m$ *leaving* K *invariant. Let P be a G-structure on M and K the tensor field on M defined by* K *and P. Then*

1. *a diffeomorphism* $\varphi : M \to M$ *is an automorphism of the G-structure P iff* φ *leaves K invariant*;
2. *a vector field* ξ *on M is an infinitesimal automorphism of P iff* $£_{L\xi} K = 0$.

Now, we can use the concept of a *G*-reductive Lie derivative to state an analogous result for generalized automorphisms of *P*.

Proposition 6.19. *In the same hypotheses of the previous proposition,*

1. *an automorphism* $\Phi : LM \to LM$ *is a generalized automorphism of the G-structure P iff* Φ *leaves K invariant*;
2. *a* $\mathrm{GL}(m, \mathbb{R})$-*invariant vector field* Ξ *on LM is an infinitesimal generalized automorphism of P iff* $£_\Xi^G K = 0$.

Note that the Lie derivative $£_\Xi^G K$ is well-defined since *K* is a tensor field on *M* and therefore a section of a vector bundle associated with $W^{0,0}(LM) \cong LM$. Here, $Q = LM$ and $H = \mathrm{GL}(m, \mathbb{R})$. Nevertheless, consistently with what we said previously, *K* has to be regarded here as a section of a *gauge*-natural, not simply natural, bundle over *M* (cf. Example 4.10). The choice $P = Q = LM$, $G = H = \mathrm{GL}(m, \mathbb{R})$ and $\Xi_K = \Xi = L\xi$ reproduces Kobayashi's classical result.

Corollary 6.20. *Let* Ξ *be a generalized infinitesimal automorphism of* $\mathrm{SO}(M, g)$. *Then,* $£_\Xi^{\mathrm{SO}(p,q)} g = 0$.

The last corollary suggests that Bourguignon and Gauduchon's metric Lie derivative might be a particular instance of a reductive Lie derivative. This is precisely the case, as explained in the following fundamental example.

Example 6.21 (Lie derivative of spinor fields. II). We know that the Kosmann lift ξ_K onto $\mathrm{SO}(M, g)$ of a vector field ξ on M is an $\mathrm{SO}(p, q)$-invariant vector field on $\mathrm{SO}(M, g)$, and hence its lift $\tilde{\xi}_K$ onto $\mathrm{Spin}(M, g)$ is a $\mathrm{Spin}(p, q)$-invariant vector field. As the spinor bundle $S(M)$ is a vector bundle associated with $\mathrm{Spin}(M, g)$, the $\mathrm{SO}(p, q)$-reductive Lie derivative $\pounds_{\tilde{\xi}_K}^{\mathrm{SO}(p,q)}\psi$ of a spinor field ψ coincides with $\pounds_\xi \psi$, i.e. locally with expression (6.5) or (6.5′). Indeed, in this case we have, with an obvious notation, $Q = P = \mathrm{SO}(M, g)$, $H = G = \mathrm{SO}(M, g)$, $\tilde{Q} = \mathrm{Spin}(M, g)$ and $\tilde{Q}_{\tilde{\lambda}} = S(M)$.

For $\pounds_{\xi_K}^{\mathrm{SO}(p,q)} g$ a similar remark to the one above for $\pounds_\Xi^G K$ applies and therefore, if $g = g_{\mu\nu}\, \mathrm{d}x^\mu \vee \mathrm{d}x^\nu$ in some natural chart, we have the local expression

$$\pounds_{\xi_K}^{\mathrm{SO}(p,q)} g_{\mu\nu} \equiv \xi^\rho \partial_\rho g_{\mu\nu} + 2g_{\rho(\mu}(\xi_K)_{\nu)}^\rho$$

$$\equiv \xi^\rho \partial_\rho g_{\mu\nu} + g_{\rho(\mu}\partial_{\nu)}\xi^\rho - \delta_{(\mu}^\rho g_{\nu)\sigma}\partial_\rho \xi^\sigma - \xi^\rho \delta_{(\mu|}^\sigma \partial_\rho g_{|\nu)\sigma} \equiv 0 \equiv \pounds_\Xi^{\mathrm{SO}(p,q)} g_{\mu\nu}$$

quite different from the usual (*natural*) Lie derivative

$$\pounds_{L\xi} g_{\mu\nu} \equiv \xi^\rho \partial_\rho g_{\mu\nu} + 2g_{\rho(\mu}(L\xi)_{\nu)}^\rho \equiv \xi^\rho \partial_\rho g_{\mu\nu} + 2g_{\rho(\mu}\partial_{\nu)}\xi^\rho$$

$$\equiv 2\nabla_{(\mu}\xi_{\nu)} \equiv \pounds_\xi g_{\mu\nu} \equiv \pounds_\Xi g_{\mu\nu}.$$

7. Discussion

In this paper we have investigated the hoary problem of the Lie derivative of spinor fields from a very general point of view, following a functorial approach. We have done so by relying on three nice geometric constructions: split structures, gauge-natural bundles and the general theory of Lie derivatives.

Such analysis has shown that, although for (purely) natural objects over a manifold M there is a conceptually and mathematically natural definition of a Lie derivative with respect to a vector field on M, there is no such thing for more general gauge-natural objects, the vector field on M being necessarily replaced by a G-invariant vector field on some principal bundle $P(M, G)$.

Conceptually speaking, though, there are at least two obvious definitions of a Lie derivative of spinor fields, both relying on a canonical, not natural, lift of a vector field on M onto the bundle of its orthonormal frames, the so-called "Kosmann lift". Both definitions are geometrically well-defined and have their own range of applicability, but, in general, only the gauge-natural one reduces to the standard definition of a Lie derivative on natural objects.

Acknowledgements

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